Properties of Rational Functions

A ***rational function*** is a function of the form $R(x)=\frac{p(x)}{q(x)}$, where p and q are polynomial functions and q is not the zero polynomial.

Finding the Domain of a Rational Function
The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Examples: Find the domain of each rational function.

a) $R(x)=\frac{2x^2-3}{x+3}$

$x+3\neq 0$

$\boxed{\{x\neq -3\}}$

b) $R(x)=\frac{x+3}{x^2-9}=\frac{x+3}{(x+3)(x-3)}$

$(x+3)(x-3)\neq 0$

$\boxed{\{x \mid x\neq -3,3\}}$

c) $R(x)=\frac{x^2}{x^2+7x+12}=\frac{x^2}{(x+3)(x+4)}$

$(x+3)(x+4)=0$

$\boxed{\{x \mid x\neq -3,-4\}}$

★ **Note:** It is important to understand that $R(x)=\frac{x+3}{x^2-9}$ and $f(x)=\frac{1}{x-3}$ are not the same. They have different domains. Their graphs are nearly identical, but the graph of the first function has a hole in it at $x=-3$, while the graph of the second function does not.

A rational function $R(x)=\frac{p(x)}{q(x)}$ is in ***lowest terms*** if $p(x)$ and $q(x)$ have no common factors.

Finding the Intercepts of a Rational Function
★ **Warning:** Always write the rational function in lowest terms before finding the x-intercepts of the graph. Otherwise, you may end up listing values that are actually holes in the graph. When finding the y-intercepts, remember that if zero is not in the domain, there is no y-intercept

To find the x-intercepts (real zeros) of the graph of a rational function, we set $R(x)=0$ and solve for x.

Notice that if $R(x)=\frac{p(x)}{q(x)}=0$, $p(x)$ must equal zero. It is the numerator that tells us about the x-intercepts.

- To find the ***x*-intercepts** (real zeros), simplify the rational function and set the numerator equal to zero.
- To find the ***y*-intercept,** plug $x=0$ into either the simplified or the unsimplified function. If zero is not in the domain of the function, there is no x-intercept.

Examples: Find the x- and y- intercepts of each rational function.

a) $R(x)=\frac{x+2}{x^2-16}=\frac{x+2}{(x+4)(x-4)}$

x-int: $x+2=0$
$x=-2$ $\boxed{(-2,0)}$

y-int: $R(0)=\frac{0+2}{0^2-16}=-\frac{1}{8}$ $\boxed{(0,-\frac{1}{8})}$

b) $R(x)=\frac{x^2-2x-35}{x^2+11x+30}=\frac{(x-7)\cancel{(x+5)}}{(x+6)\cancel{(x+5)}}=\frac{x-7}{x+6}$

x-int: $x-7=0$
$x=7$ $\boxed{(7,0)}$

y-int: $R(0)=\frac{0-7}{0+6}=-\frac{7}{6}$ $\boxed{(0,-\frac{7}{6})}$

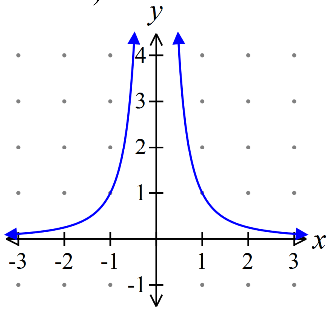
c) $R(x)=\frac{x}{x^2+8x}=\frac{\cancel{x}}{\cancel{x}(x+8)}=\frac{1}{x+8}$

$\boxed{\text{no x-int}}$ (1 is never equal to 0)

Domain: $\{x \mid x\neq 0,-8\}$
zero is not in the domain.

$\boxed{\text{no y-int}}$

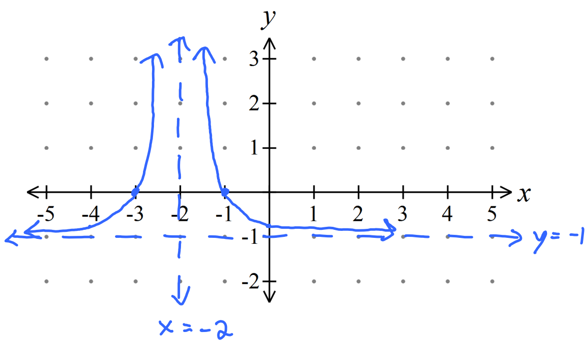
Example: Analyze the graph of $H(x)=\frac{1}{x^2}$. (Find the domain, range, intercepts, and any other important features).



Domain: $(-\infty,0)\cup(0,\infty)$ or $\{x \mid x\neq 0\}$
Range: $(0,\infty)$ or $\{y \mid y>0\}$
No intercepts
Symmetric around y-axis (even)

Example: Using transformations, graph $R(x)=\frac{1}{(x+2)^2}-1$.

$\leftarrow 2 \quad \downarrow 1$



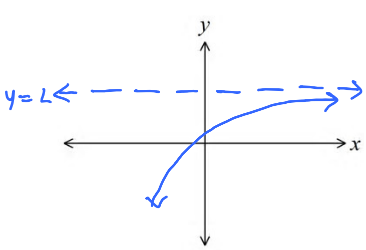
Asymptotes
In the previous example, notice that as the values of x become more negative, that is, as x becomes ***unbounded in the negative direction*** $(x\to-\infty$, read "as x approaches negative infinity"$)$ the values of $R(x)$ approach -1. We conclude the following:

1. As $x\to-\infty$, the values of $R(x)\to-1$.
2. As $x\to-2$, the values of $R(x)\to\infty$.
3. As $x\to\infty$, the values of $R(x)\to-1$.

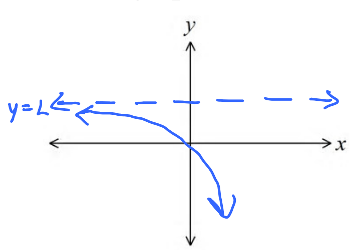
The graph goes near, but does not reach the lines $x=-2$ and $y=-1$. These lines are called ***asymptotes.***

Let R denote a function.

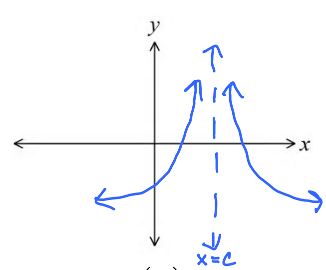
- If, as $x\to-\infty$ or as $x\to\infty$, the values of $R(x)$ approach some fixed number L, then the line $y=L$ is a ***horizontal asymptote*** of the graph of R.
- If, as x approaches some number c, the values of $R(x)$ approach ∞ or $-\infty$, then the line $x=c$ is a ***vertical asymptote*** of the graph of R.



$\lim_{x\to\infty} R(x)=L$

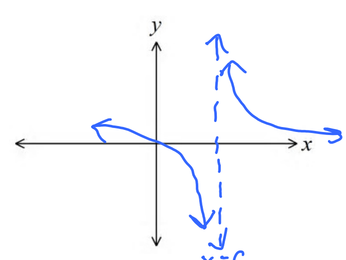


$\lim_{x\to-\infty} R(x)=L$



$\lim_{x\to c^-} R(x)=\infty$

$\lim_{x\to c^+} R(x)=\infty$



$\lim_{x\to c^-} R(x)=-\infty$

$\lim_{x\to c^+} R(x)=\infty$

There is a third type of asymptote called an ***oblique asymptote.*** An oblique asymptote occurs when the graph's end behavior follows an oblique line (a linear function of the form $y = ax + b,\ a \neq 0$). A graph may also approach some function such as a parabola.

★ **Note:** The graph of a function may intersect a horizontal or oblique asymptote at some other point, but the graph will never intersect a vertical asymptote.

Find the Vertical Asymptotes of a Rational Function

A rational function $R(x) = \frac{p(x)}{q(x)}$, in *lowest terms* will have a vertical asymptote at $x = r$ if r is a real zero of the denominator q. That is, if $x - r$ is a factor of the denominator q of the rational function in lowest terms, it will have the vertical asymptote $x = r$.

- **Warning:** If the rational function is not in lowest terms, this theorem will result in an incorrect listing of vertical asymptotes.

Example: Find the vertical asymptotes, if any, of the graph of each rational function.

a) $R(x) = \frac{x+1}{x^2 - 9} = \frac{x+1}{(x+3)(x-3)}$

$(x+3)(x-3) = 0$

$\boxed{x = -3,\ x = 3}$

b) $R(x) = \frac{x+2}{x^2 - 3x - 10} = \frac{\cancel{x+2}}{(x-5)\cancel{(x+2)}} = \frac{1}{x-5}$

$x - 5 = 0$

$\boxed{x = 5}$

c) $R(x) = \frac{x}{x^3 + 10x^2 + 9x} = \frac{x}{x(x^2+10x+9)}$

$= \frac{\cancel{x}}{\cancel{x}(x+9)(x+1)} = \frac{1}{(x+9)(x+1)}$

$(x+9)(x+1) = 0$

$\boxed{x = -9,\ x = -1}$

d) $R(x) = \frac{x^2}{x^2 + 4}$ ← prime

$x^2 + 4 = 0$

$x^2 = -4$

$x = \pm\sqrt{-4} = \pm 2i$

$\boxed{\text{no vertical asymptotes}}$

Find the Horizontal or Oblique Asymptotes of a Rational Function

- If a rational function $R(x)$ is ***proper,*** that is *if the degree of the numerator is less than the degree of the denominator*, then the line $y = 0$ is a horizontal asymptote.
- If a rational function $R(x) = \frac{p(x)}{q(x)}$ is ***improper,*** that is, if the *degree of the numerator is greater than or equal to the degree of the denominator*, we can use long division to write the rational function as the sum of a polynomial $f(x)$ plus the proper rational function $\frac{r(x)}{q(x)}$. As $x \to -\infty$ or as $x \to \infty$, $\frac{r(x)}{q(x)} \to 0$, and the graph of $R(x)$ approaches the graph of $f(x)$.
  - The possibilities include:
    1. If $f(x) = b$, a constant, the line $y = b$ is a horizontal asymptote of the graph of R.
    2. If $f(x) = ax + b,\ a \neq 0$, the line $y = ax + b$ is an oblique (slanted) asymptote of the graph of R.
    3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

Examples: Find the horizontal or oblique asymptotes, if any, of the graph of the function.

a) $R(x)=\dfrac{x+5}{x^2-3x-10}$ ← Deg 1 / ← Deg 2

Deg num < Deg denom

$\boxed{\text{Horizontal asymptote } y=0}$

b) $H(x)=\dfrac{6x^2-2x+5}{3x^2-2}$ ← Deg 2 / ← Deg 2

Deg num = Deg denom

$y=\frac{6}{3}$ $\boxed{\text{Horizontal asymptote at } y=2}$

c) $H(x)=\dfrac{8x^3+2x^2-6}{2x^2-3}$ ← Deg 3 / Deg 2

Deg num > Deg denom

$$\begin{array}{r l}
 & 4x+1 \\
2x^2-3 \,\big)\!\! & 8x^3+2x^2+0x-6 \\
 & -(8x^3 \qquad\quad -12x) \\
 & \qquad 2x^2+12x-6 \\
 & \quad -(2x^2 \qquad\quad -3) \\
 & \qquad\qquad 12x-3
\end{array}$$

$H(x)=4x+1+\frac{12x-3}{2x^2-3}$

$\boxed{\text{oblique asymptote: } y=4x+1}$

d) $H(x)=\dfrac{3x^4+4x^2-7}{x^2-2x+5}$ ← Deg 4 / Deg 2

Deg num > Deg denom

$$\begin{array}{r l}
 & 3x^2+6x+1 \\
x^2-2x+5 \,\big)\!\! & 3x^4+0x^3+4x^2+0x-7 \\
 & -(3x^4-6x^3+15x^2) \\
 & \qquad 6x^3-11x^2+0x \\
 & \quad -(6x^3-12x^2+30x) \\
 & \qquad\qquad x^2-30x-7 \\
 & \qquad\quad -(x^2-2x+5) \\
 & \qquad\qquad -28x-12
\end{array}$$

$H(x)=3x^2+6x+1+\frac{-28x-12}{x^2-2x+5}$

$\boxed{\text{No horizontal or oblique asymptotes. } y=3x^2+6x+1 \text{ acts like an asymptote.}}$

Summary:

Given a rational function $R(x) = \dfrac{p(x)}{q(x)}$,

1. The **domain** is found by setting the denominator not equal to zero and solving. (Do not simplify first).
2. The ***x*-intercepts** are found by setting the numerator to zero and solving. (Simplify first).
3. The ***y*-intercept** is found by setting $x = 0$ and simplifying. (You can do this before or after simplifying the function). Remember, if zero is not in the domain, the function has no *y*-intercept.
4. **Vertical asymptotes** are found by setting the denominator to zero and solving (Simplify first).
5. **Horizontal or Oblique Asymptotes:**
   a) If **proper,** where $\textit{degree of numerator} < \textit{degree of denominator}$, the graph has a **horizontal asymptote** at $y = 0$.
   b) If **improper**, where $\textit{degree of numerator} = \textit{degree of denominator}$, the graph has a **horizontal asymptote** at the line $y = \dfrac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
   c) If **improper**, where $\textit{degree of numerator} > \textit{degree of denominator (by one)}$, the graph has an **oblique asymptote** of $y = f(x)$ found by performing long division.
   d) If **improper**, where $\textit{degree of numerator} > \textit{degree of denominator (by more than one)}$, then *R* has neither a horizontal nor an oblique asymptote, but the end behavior can be determined using long division.